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Hypersonic Flow over an Oscillating Wedge

S. C. McIntosh Jr.*
Stanford University, Stanford, Calif.

The equations of hypersonic small-disturbance theory are perturbed for small oscillations of a thin, semi-infinite, flexible wedge. A solution for arbitrary smooth mode shapes is found in terms of infinite series. These series express the effects of acoustic waves, generated by the unsteady motion, that propagate in the flow field between the bow shock wave and the wedge surface. They reflect from the wedge surface and the shock wave, but they are attenuated only at the shock wave. Thus the degree of attenuation determines the influence of the waves. This attenuation decreases as the shock strength increases. It is then shown that the results from more approximate methods, such as piston theory and others that attempt to account for a strong bow shock, can be reproduced by successively neglecting terms in the "complete" solution. It appears that the most significant contribution of the complete solution is to change the phase shift of the unsteady pressure. Finally, it is shown that the complete solution is the only one that will produce an adequate representation in the double limit of very large Mach number and adiabatic exponent near unity.

Introduction

HYPERSONIC flow is distinguished by the fact that the governing equations are nonlinear even for the flow over slender bodies. From the viewpoint of unsteady aerodynamics, this means that the body cannot be idealized to a surface of zero thickness; it will create a shock wave whose effects on the ensuing flow must be considered.

Piston theory has been widely used as a first step in approximating such effects. The theoretical basis for piston theory has been known for many years, but it is generally accepted that Lighthill[1] was the first to recognize its utility. The theoretical and practical limitations on its applicability to aeroelastic problems have been studied by Ashley and Zartarian,[2] among others. The basis for using a piston formula in hypersonic flow is provided by the unsteady analogy of hypersonic small-disturbance theory.[3,4] It is found that the reduced equations for this theory will transform directly to those for unsteady flow in the transverse plane. For example, we may replace a two-dimensional steady or unsteady problem by that of a one-dimensional piston moving in a tube filled with a gas. The velocity of the piston is identified with the velocity imparted to the fluid by the body. If the original problem is unsteady, there is an additional term due to the motion of the body. However, the character of the piston problem is unchanged, and so in principle there is no difference between steady and unsteady flow in hypersonic small-disturbance theory. In general, the piston will generate at least one shock wave, which corresponds to the bow shock wave.

The problem is greatly simplified by assuming that the piston generates only acoustic waves, for then there exists a very simple relation between the velocity of the piston and the pressure on the face of the piston which corresponds to the surface pressure. The pressure thus calculated gives excellent results provided a number of conditions are met; the details are in Refs. 1 and 2. Physically, this approximation does not take into account three phenomena: the effect of the bow shock wave on the flow, the effect of any interaction between subsequent waves and the bow shock wave, and the effect of waves stronger than acoustic generated by the unsteady motion of the body. For most purposes, the third restriction is not serious, since it is equivalent to linearization in the amplitude of the unsteady motion; indeed, this is all

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* Research Assistant, Department of Aeronautics and Astronautics. Member AIAA.

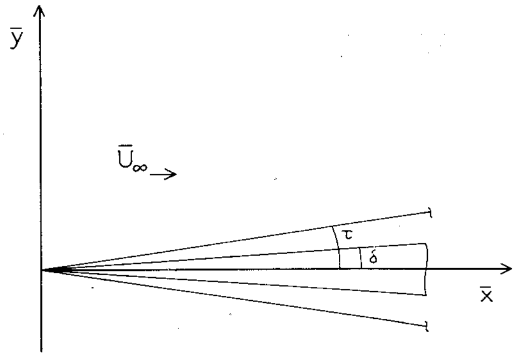

Fig. 1 Coordinate system.

that is usually desired for an aeroelastic stability analysis, such as a flutter analysis. The other two restrictions may or may not be serious. However, it is evident that as the Mach number increases the effect of the bow shock wave will become significant.

As a next step, one might then consider perturbing the steady flow behind the shock wave, rather than the freestream, for the unsteady motion. We illustrate this by referring to a paper by Miles.[5] He writes the plane-wave relation for small unsteady perturbations of a steady flow as

$$\bar{p}' = \overline{\rho a v}' \tag{1}$$

where $\bar{p}'$ and $\bar{v}'$ are, respectively, the perturbation pressure and the perturbation velocity normal to the wavefront, and $\bar{\rho}$ and $\bar{a}$ are, respectively, the local density and sonic speed. The usual piston-theory formula is obtained from this by expanding the acoustic relations to second order to obtain $\bar{\rho}$ and $\bar{a}$. We can correct more accurately for the shock wave by using the shock relations written for hypersonic small-disturbance theory.[4, 5] However, these relations give not only $\bar{\rho}$ and $\bar{a}$, but also $\bar{p}$, the pressure, as a function of the instantaneous piston velocity. Thus we can calculate $\bar{\rho}$ and $\bar{a}$ by using only that part of the piston velocity due to the slope of the body, or we can add the term due to the unsteady motion of the body and substitute this in the pressure shock relation. Miles illustrates the difference between these two choices by using the "strong-shock" relations. If we use them to calculate $\bar{\rho}$ and $\bar{a}$, we obtain from Eq. (1)

$$\bar{p}' = (\gamma + 1)\left[\frac{\gamma}{2(\gamma - 1)}\right]^{1/2} \bar{\rho}_\infty \bar{a}_\infty M_\infty \delta \bar{v}' \tag{2}$$

where γ is the adiabatic exponent, M_∞ the freestream Mach number, and δ the local steady surface slope. The subscript ∞ refers to freestream conditions. We note that the assumptions used to derive this expression become questionable as $\gamma \to 1$. If, on the other hand, we use the pressure shock relation with both terms for the velocity and linearize in $\bar{v}'$, we obtain

$$\bar{p}' = (\gamma + 1)\bar{\rho}_\infty \bar{a}_\infty M_\infty \delta \bar{v}' \tag{3}$$

The unsteady pressure from Eq. (2) is larger than that from Eq. (3), and the difference increases as $\gamma \to 1$. Miles suggested that Eq. (3), being more of a "tangent-wedge" approximation, would be better for motions involving the body as a whole, whereas Eq. (2) would be better for motions involving small portions of the body.

In addition, Zartarian, Hsu, and Ashley[6] suggested employing the unsteady analogy in reverse to obtain an equivalent problem in steady flow. Thus an unsteady problem would be reduced to a steady problem, with the body shape distorted slightly to account for the unsteady motion. The surface pressure is then found with the shock-expansion method.[7, 8] The heart of this approximation is that waves generated by the irregularities on the surface (as well as by the basic surface) are strongly attenuated at the bow shock wave and thus that their downstream effects are negligible.

Finally, as the Mach number becomes very large, we enter the domain of Newtonian theory, characterized by the double limit $M_\infty \to \infty$, $\gamma \to 1$.

We propose to find a solution to an unsteady problem which will permit us to compare these theories on a systematic basis. This solution will still contain only the linear effects of the unsteady motion, but it will take into account all the other effects that were neglected. In particular, we will comment on Miles' conjecture, examine the attenuation of waves at the bow shock, and pursue further the relationship between a solution in hypersonic flow and its counterpart under Newtonian conditions.

The choice of a wedge is dictated primarily by the simplicity of the basic flow. Also, we note that Carrier[9, 10] used the wedge to study unsteady thickness effects in supersonic flow and that this solution was extended by Van Dyke.[11]

Hypersonic Small-Disturbance Theory

General Problem and Solution

Let $\bar{x}$ and $\bar{y}$ be the coordinates in a Cartesian coordinate system aligned with a thin semi-infinite wedge as shown in Fig. 1. The velocity $\bar{U}_\infty$ of a uniform inviscid freestream of a perfect gas is aligned with the $\bar{x}$ axis. The slope of the steady bow shock wave is τ; the corresponding wedge slope is $\delta = b\tau$. The time, the velocity in the $\bar{x}$ and $\bar{y}$ directions, the pressure, density, Mach number, and ratio of specific heats are denoted, respectively, by $\bar{t}$, $\bar{u}$, $\bar{v}$, $\bar{p}$, $\bar{\rho}$, M, and γ; the subscript ∞ indicates freestream values. In accordance with Ref. 4, we introduce the following new (unbarred) independent and dependent variables in the usual unsteady equations of continuity, momentum, and entropy:

$$\begin{gathered}
\bar{u}(\bar{x}, \bar{y}, \bar{t}) = \bar{U}_\infty[1 + \tau^2 u(x, y, t)] \\
\bar{v}(\bar{x}, \bar{y}, \bar{t}) = \bar{U}_\infty \tau v(x, y, t) \\
\bar{\rho}(\bar{x}, \bar{y}, \bar{t}) = \bar{\rho}_\infty \rho(x, y, t) \\
\bar{p}(\bar{x}, \bar{y}, \bar{t}) = \bar{p}_\infty \gamma M_\infty{}^2 \tau^2 p(x, y, t) \\
\bar{x} = x \qquad \bar{y} = \tau y \qquad \bar{t}\bar{U}_\infty = t
\end{gathered} \tag{4}$$

The unbarred variables are assumed of order unity. When terms of $0(\tau^2)$ or smaller are neglected, the resultant equations are

$$\begin{gathered}
\rho_t + \rho_x + (\rho v)_y = 0 \text{ (continuity)} \\
v_t + v_x + v v_y + p_y/\rho = 0 \text{ (}y\text{ momentum)} \\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\frac{p}{\rho^\gamma} = 0 \text{ (entropy)} \\
u_t + u_x + v u_y + p_x/\rho = 0 \text{ (}x\text{ momentum)}
\end{gathered} \tag{5}$$

The letter subscripts signify partial differentiation. The x momentum equation is decoupled from the others and is not considered further.

We will describe the upper surface of the oscillating wedge as the real part of †

$$\bar{y}_w = \bar{x} b \tau - \bar{\alpha} e^{i\omega \bar{t}} f_w(\bar{x}) \tag{6}$$

where $\bar{\alpha}$ is the amplitude of the oscillation, ω its frequency, f_w its mode shape, and $i = (-1)^{1/2}$. It is important to note that f_w may be used to represent the oscillation of a rigid or a flexible wedge. We assume that f_w and f_w' are of order unity, and that $\bar{\alpha}$ is small. The lower surface is found by reversing the sign of $\bar{\alpha}$. We now restrict ourselves to the upper sur-

† It will be understood that throughout this paper we consider the real part of all complex expressions.

face, noting that the solution for the lower surface can be obtained simply by changing this sign. In reduced variables, the upper surface becomes

$$y_w = bx - \alpha e^{ikt} f_w(x) \tag{7}$$

where $k = \omega/\bar{U}_\infty$ and $\alpha = \bar{\alpha}/\tau$. We assume k of order unity, and we assume that α is small, so that the actual amplitude $\bar{\alpha}$ must be small compared with τ. Similarly, we describe the shock wave as

$$y_s = x - \alpha e^{ikt} f_s(x) \tag{8}$$

The unknown shock-wave perturbation f_s and its derivative are also assumed of order unity.

The dependent variables are perturbed as follows:

$$\begin{aligned} p(x, y, t) &= p_0 + \alpha e^{ikt} p_1(x, y) \\ \rho(x, y, t) &= \rho_0 + \alpha e^{ikt} \rho_1(x, y) \\ v(x, y, t) &= v_0 + \alpha e^{ikt} v_1(x, y) \end{aligned} \tag{9}$$

These expressions are substituted into the first three equations of Eqs. (5). Retaining only terms of $0(\alpha)$, we obtain

$$\begin{aligned} &\left(ik + \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}\right) \rho_1 + \rho_0 v_{1y} = 0 \\ &\left(ik + \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}\right) \frac{v_1 + p_{1y}}{\rho_0} = 0 \\ &\left(ik + \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}\right) (\rho_0 p_1 - \gamma p_0 \rho_1) = 0 \end{aligned} \tag{10}$$

These equations can be greatly simplified by further transformation of the independent and dependent variables. For the independent variables, we write

$$\eta = y - v_0 x \qquad \xi = x \tag{11}$$

Since lines $y - v_0 x = \text{const}$ are streamlines in the unperturbed flow behind the shock wave, we see that this is a von Mises transformation. For the dependent variables, we write

$$\begin{aligned} v_1(x, y) &= e^{-ik\xi} V_1(\xi, \eta) \\ p_1(x, y) &= e^{-ik\xi} P_1(\xi, \eta) \\ \rho_1(x, y) &= e^{-ik\xi} R_1(\xi, \eta) \end{aligned} \tag{12}$$

These transformations are then applied to Eqs. (10). We find

$$R_{1\xi} + \rho_0 V = 0 \tag{13a}$$

$$V_{1\xi} + P_{1\eta}/\rho_0 = 0 \tag{13b}$$

$$(\partial/\partial\xi)(\rho_0 P_1 - \gamma p_0 R_1) = 0 \tag{13c}$$

We use Eq. (13c) to eliminate $R_{1\xi}$ in Eq. (13a), thereby obtaining two equations for P_1 and V_1. These two equations are then cross-differentiated and combined to give separate equations for P_1 and V_1

$$\begin{aligned} V_{1\xi\xi} - a_0^2 V_{1\eta\eta} &= 0 \\ P_{1\xi\xi} - a_0^2 P_{1\eta\eta} &= 0 \end{aligned} \tag{14}$$

where $a_0^2 = \gamma p_0/\rho_0$. These equations are readily integrable in terms of four arbitrary functions. Their number can be reduced to two by substituting the solutions for P_1 and V_1 into Eq. (13b) and comparing functions of the same independent variable. Finally, R_1 is obtained by integrating Eq. (13c). Thus we have in summary

$$\begin{aligned} V_1 &= f_v(\eta - a_0\xi) + g_v(\eta + a_0\xi) \\ P_1 &= (\gamma p_0 \rho_0)^{1/2} [f_v(\eta - a_0\xi) - g_v(\eta + a_0\xi)] \\ R_1 &= [f_v(\eta - a_0\xi) - g_v(\eta + a_0\xi)]\rho_0/a_0 + S(\eta) \end{aligned} \tag{15}$$

The unknown function $S(\eta)$ comes from integrating Eq. (13c). S, f_v, g_v, and the unknown shock shape f_s are the four functions that are to be determined from three shock relations plus a tangency condition at the wedge surface.

The shock relations are obtained by perturbing Eqs. (22) of Ref. 4. These equations give the pressure, velocity, and density at the shock wave in terms of γ and the hypersonic similarity parameter based on the local shock-wave slope. For the latter we substitute the expression

$$M_\infty \tau \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) y_s = M_\infty \tau [1 - \alpha e^{ikt}(f_s' + ikf_s)] \tag{16}$$

The pressure, velocity, and density at the shock wave are given to $0(\alpha)$ by evaluating them at the unperturbed shock position $y = x$. The results are

$$\begin{aligned} p &= \frac{2\gamma K^2 - (\gamma - 1)}{\gamma(\gamma + 1)K^2} - \frac{4}{\gamma + 1} \alpha e^{ikt}(f_s' + ikf_s) \\ v &= \frac{2(K^2 - 1)}{(\gamma + 1)K^2} - \frac{2(K^2 + 1)}{(\gamma + 1)K^2} \alpha e^{ikt}(f_s' + ikf_s) \\ \rho &= \frac{(\gamma + 1)K^2}{2 + (\gamma - 1)K^2} \left[1 - \frac{4}{2 + (\gamma - 1)K^2} \times \alpha e^{ikt}(f_s' + ikf_s)\right] \end{aligned} \tag{17}$$

at $y = x$. $K = M_\infty \tau$ is the hypersonic similarity parameter based on the unperturbed shock-wave slope. The tangency condition is given by

$$v = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) y_w = b - \alpha e^{ikt}(f_w' + ikf_w) \text{ at } y = bx \tag{18}$$

We note that p_0, $v_0(=b)$, and ρ_0 are given by the leading terms in Eqs. (17). The boundary conditions for the perturbation quantities are then found by transforming according to Eqs. (11) and (12)

$$\left.\begin{aligned} P_1 &= -\frac{4}{\gamma + 1} f_s^*(\xi) \\ V_1 &= -\frac{2(K^2 + 1)}{(\gamma + 1)K^2} f_s^*(\xi) \\ R_1 &= -\frac{4\rho_0}{2 + (\gamma - 1)K^2} f_s^*(\xi) \end{aligned}\right\} \text{ at } \eta = (1 - b)\xi \tag{19}$$

$$V_1 = -f_w^*(\xi) \text{ at } \eta = 0$$

where

$$\begin{aligned} f_s^*(\xi) &= (d/d\xi)[e^{ik\xi} f_s(\xi)] \\ f_w^*(\xi) &= (d/d\xi)[e^{ik\xi} f_w(\xi)] \end{aligned} \tag{20}$$

Substituting then for P_1, V_1, and R_1, we find

$$\begin{aligned} f_v(-a_0\xi) + g_v(a_0\xi) &= -f_w^*(\xi) \\ f_v(-A\xi) - g_v(B\xi) &= -Df_s^*(\xi) \\ f_v(-A\xi) + g_v(B\xi) &= -Cf_s^*(\xi) \\ (\rho_0/a_0)[f_v(-A\xi) - g_v(B\xi)] + S[\xi(1 - b)] &= -Ef_s^*(\xi) \end{aligned} \tag{21}$$

where

$$\begin{aligned} A &= -(1 - b - a_0) \\ B &= 1 - b + a_0 \\ C &= \frac{2(K^2 + 1)}{(\gamma + 1)K^2} \\ D &= \frac{4}{(\gamma + 1)(\gamma p_0 \rho_0)^{1/2}} \\ E &= \frac{4\rho_0}{2 + (\gamma - 1)K^2} \end{aligned} \tag{22}$$

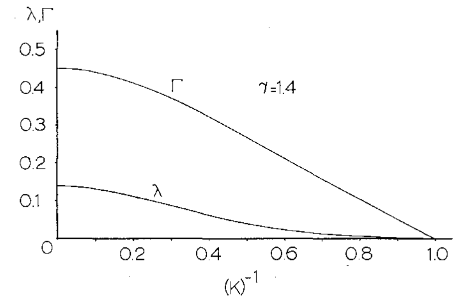

Fig. 2 Attenuation factor and reflection coefficient vs inverse of hypersonic similarity parameter.

By manipulating the first three equations of Eqs. (21), we obtain a functional equation for f_v alone:

$$f_v(\xi/\Gamma) + \lambda f_v(\xi) = F_w^*(\xi/\Gamma) \tag{23}$$

where

$$\Gamma = A/B$$
$$\lambda = (C - D)/(C + D) \tag{24}$$
$$F_w^*(\xi) = -f_w^*(-\xi/a_0)$$

The parameters λ and Γ are both positive and smaller than unity, as can be seen in Fig. 2, where they are plotted vs $1/K$ for $\gamma = 1.4$.

Equation (23) is similar to one solved by Chu,[12] who was interested in the steady interaction of Mach waves with strong shock waves. The general solution of this equation is a particular solution plus any multiple of the homogeneous solution. Following Chu, we write the particular solution as

$$f_{vp}(\xi) = F_w^*(\xi) - \lambda F_w^*(\Gamma\xi) + \lambda^2 F_w^*(\Gamma^2\xi) - \ldots = \sum_{n=0}^{\infty} (-\lambda)^n F_w^*(\Gamma^n\xi) \tag{25}$$

As $n \to \infty$, both $(-\lambda)^n$ and Γ^n approach zero. F_w^*, related through Eqs. (24) and (20) to f_w, is of order unity. Thus it is clear that this series is convergent.

To obtain the equation for the homogeneous solution, we rearrange Eq. (23) with the right-hand side zero:

$$f_{vh}(\Gamma\xi) = -(1/\lambda)f_{vh}(\xi) \tag{26}$$

This equation can be interpreted as a recursion formula. Given f_{vh} at some point ξ, we can calculate it at a number of other discrete points $\Gamma^n\xi$ by the formula

$$f_{vh}(\Gamma^n\xi) = (-\lambda)^{-n} f_{vh}(\xi) \tag{27}$$

As $n \to \infty$, the sequence $\Gamma^n\xi$ approaches zero, and for any nonzero value of $f_{vh}(\xi)$, the sequence $(-\lambda)^{-n}f_{vh}(\xi)$ diverges. Therefore we must set the constant multiplying the homogeneous solution equal to zero, since we reject any solution that diverges at the vertex of the wedge.

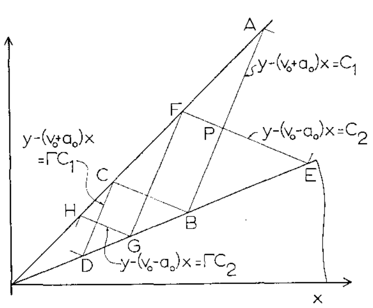

Fig. 3 Propagation of disturbances along characteristic lines.

The remaining functions g_v, S, and f_s^* are found by further manipulation of Eqs. (21); then f_s^* is integrated to give f_s. We find

$$g_v(\xi) = -\sum_{n=1}^{\infty} (-\lambda)^n F_w^*(-\Gamma^n\xi)$$

$$f_s(\xi) = \frac{2e^{-ik\xi}}{C+D}\sum_{n=0}^{\infty} (-\lambda)^n e^{ik\Gamma^n(A/a_0)\xi} f_w\left(\Gamma^n \frac{A}{a_0}\xi\right) - \frac{f_w(0)}{C} \tag{28}$$

$$S(\eta) = -\frac{8\rho_0}{C+D} \times \left\{\frac{(\gamma-1)K^4 + 2(\gamma+1)K^2 - (\gamma-1)}{[2+(\gamma-1)K^2][2\gamma K^2 - (\gamma-1)]}\right\} \times \sum_{n=0}^{\infty} (-\lambda)^n F_w^*\left(-\Gamma^n\frac{\eta A}{1-b}\right)$$

We can now summarize the perturbation flow quantities and shock-wave shape in terms of the variables of hypersonic small-disturbance theory:

$$p_1(x,y) = -e^{-ikx}(\gamma p_0\rho_0)^{1/2} \times \left\{\sum_{n=0}^{\infty}(-\lambda)^n f_w^*\left[-\frac{\Gamma^n}{a_0}(y - v_0x - a_0x)\right] + \sum_{n=1}^{\infty}(-\lambda)^n f_w^*\left[\frac{\Gamma^n}{a_0}(y - v_0x + a_0x)\right]\right\}$$

$$v_1(x,y) = -e^{-ikx} \times \left\{\sum_{n=0}^{\infty}(-\lambda)^n f_w^*\left[-\frac{\Gamma^n}{a_0}(y - v_0x - a_0x)\right] - \sum_{n=1}^{\infty}(-\lambda)^n f_w^*\left[\frac{\Gamma^n}{a_0}(y - v_0x + a_0x)\right]\right\}$$

$$\rho_1(x,y) = -e^{-ikx}\frac{\rho_0}{a_0} \times \left\{\sum_{n=0}^{\infty}(-\lambda)^n f_w^*\left[-\frac{\Gamma^n}{a_0}(y - v_0x - a_0x)\right] + \sum_{n=1}^{\infty}(-\lambda)^n f_w^*\left[\frac{\Gamma^n}{a_0}(y - v_0x + a_0x)\right]\right\} + Ge^{-ikx}\sum_{n=0}^{\infty}(-\lambda)^n f_w^*\left[\Gamma^n\frac{(y - v_0x)A}{a_0(1-v_0)}\right] \tag{29}$$

$$f_s(x) = e^{-ikx}\left[\frac{2}{C+D} \times \sum_{n=0}^{\infty}(-\lambda)^n e^{ik\Gamma^n(A/a_0)x} f_w\left(\Gamma^n\frac{A}{a_0}x\right) - \frac{f_w(0)}{C}\right]$$

$$G = \frac{8\rho_0}{C+D}\left\{\frac{(\gamma-1)K^4 + 2(\gamma+1)K^2 - (\gamma-1)}{[2+(\gamma-1)K^2][2\gamma K^2 - (\gamma-1)]}\right\}$$

Of primary interest are the unsteady forces and moments on the wedge. Hence we calculate as well the difference between the pressure coefficients on the lower and upper surfaces, which is the lift coefficient per unit chord. We recall that the solution for the lower surface is given by changing the sign of α. Thus we find

$$C_{pl} - C_{pu} = 4\tau^2\alpha e^{ikt}p_1(v_0x, x) \tag{30}$$

Substituting from Eqs. (29) gives finally

$$C_{pl} - C_{pu} = 4\tau\bar{\alpha}e^{ikt}\left[\frac{2\gamma K^2 - (\gamma-1)}{2+(\gamma-1)K^2}\right]^{1/2}\left\{f_w'(x) + ikf_w(x) + 2e^{-ikx}\sum_{n=1}^{\infty}(-\lambda)^n e^{ik\Gamma^n x}[f_w'(\Gamma^n x) + ikf_w(\Gamma^n x)]\right\} \tag{31}$$

The "reduced" amplitude α has been replaced by the actual amplitude $\bar{\alpha}$, the better to illustrate the dependence of $C_{pl} - C_{pu}$ on the wedge thickness and the amplitude of the unsteady motion.

Physical Interpretation

Let us now consider, as an example, the pressure p_1 as given in Eqs. (28). There are two infinite series in the expression, each of which is made up of functions constant along different families of characteristic lines. Suppose we wish to evaluate p_1 at point P in Fig. 3. The first series in the expression for p_1 from Eqs. (28) contains functions constant along lines such as $\overline{AB}$, given by $y - (v_0 + a_0)x = C_1$. The first term of the series, given by $n = 0$, represents the pressure disturbance arriving at P along $\overline{AB}$. This disturbance was caused by the wedge at B. A geometrical calculation will show that $y - (v_0 + a_0)x = \Gamma C_1$ gives the characteristic $\overline{DC}$. Thus the second term, given by $n = 1$, represents the disturbance from the wedge at D, which also arrives at P along $\overline{AB}$. However, it has been reflected once from the shock; therefore it is attenuated by a factor $(-\lambda)$. This series then represents the contribution to the perturbation pressure at P of all disturbances which arrive along $\overline{AB}$. In like manner the other series sums the contribution of all disturbances arriving along $\overline{FE}$. In accordance with acoustic theory, the disturbances propagate without attenuation except where they are reflected from the bow shock wave.

A similar interpretation applies to both v_1 and ρ_1. In addition, the equation for ρ_1 has a term that is constant along streamlines. This term takes into account the effect of the oscillation on the entropy behind the shock wave.

The effect of an incident Mach wave on a shock wave has been discussed in Refs. 7, 8, 12, and 13. Chu[12] and Chernyi[13] have derived expressions for the attenuation factor $(-\lambda)$ which reduce identically to the expression derived herein when they are written in hypersonic small-disturbance form. The expression obtained by Eggers and Syvertson[8] does not so reduce; however, it is possible to alter their interpretation slightly and produce an expression that does. Chernyi and Eggers and Syvertson have expressions more generally valid than the one we have derived, and they found that the attenuation factor can change sign. However, this change of sign occurs for either very large flow deflections or Mach numbers near unity. Both of these cases are clearly outside the domain of hypersonic small-disturbance theory.

Comparison with Other Theories

We begin by taking the second-order piston formula from Ref. 2 and calculating the difference in pressure coefficients as in Eq. (31). This gives

$$C_{pl} - C_{pu} = \frac{4\tau}{K}\,\bar{\alpha}e^{ikt}\left(1 + \frac{\gamma+1}{2}\,Kb\right)(f_w' + ikf_w) \quad (32)$$

We recall that K is the hypersonic similarity parameter based on the steady shock slope. It may also be interpreted as the Mach number of the normal shock wave caused by the analogous piston, whose Mach number is Kb. As K approaches unity, the shock wave degenerates to an acoustic wave, and b approaches zero. Hence $K = 1$ corresponds to a wedge of zero thickness, which is still in a hypersonic freestream. For such a wedge (or, in other words, for a flat plate), Eq. (32) becomes

$$C_{pl} - C_{pu} = 4\tau\bar{\alpha}e^{-ikt}(f_w' + ikf_w) \quad (33)$$

Note that τ in this formula could equally well be written as $(M_\infty)^{-1}$, since it reduces to the Mach angle in this limit.

We can take into account the existence of a strong bow shock by interpreting the uniform steady flow field between the shock and the wedge as an equivalent freestream. This freestream is now parallel to the steady surface, and so we may use the flat-plate solution, Eq. (33). We denote by M_0 the Mach number behind the shock wave in steady flow and write

$$(C_{pl} - C_{pu})_0 = (4/M_0)\,\bar{\alpha}e^{ikt}(f_w' + ikf_w) \quad (34)$$

Then we correct this expression for the difference in reference values between the equivalent freestream and the actual one. We multiply by the ratio of dynamic pressures and obtain

$$C_{pl} - C_{pu} = 4\tau\bar{\alpha}e^{ikt}\left[\frac{2\gamma K^2 - (\gamma-1)}{2 + (\gamma-1)K^2}\right]^{1/2}(f_w' + ikf_w) \quad (35)$$

We notice immediately that this is equivalent to our full solution, Eq. (31), with $\lambda = 0$, that is, when reflections from the bow wave are disregarded. In the strong-shock approximation ($K \to \infty$), Eq. (35) becomes

$$C_{pl} - C_{pu} = 4\tau\bar{\alpha}e^{ikt}[2\gamma/(\gamma-1)]^{1/2}(f_w' + ikf_w) \quad (36)$$

Let us now compare these results with Miles' formulas, Eqs. (2) and (3). We associate Miles' velocity $\bar{v}'$ with the unsteady velocity at the wedge surface from Eq. (18). From Eq. (2) we obtain for his first method

$$C_{pl} - C_{pu} = 4\tau\bar{\alpha}e^{ikt}[2\gamma/(\gamma-1)]^{1/2}(f_w' + ikf_w) \quad (37)$$

whereas from Eq. (3) we obtain for his second method

$$C_{pl} - C_{pu} = 8\tau\bar{\alpha}e^{ikt}(f_w' + ikf_w) \quad (38)$$

We see that Eqs. (36) and (37) are identical. Therefore we conclude that Miles' first method is the proper one for taking the bow shock wave into account insofar as the solution is linearized in the amplitude of the unsteady motion. We emphasize that f_w may equally well represent pitching or plunging of the wedge as a whole, a local perturbation on the wedge surface, or elastic deformation of the wedge. Our preference for Eq. (37) is based on its logical relation to Eq. (31), which gives the most complete solution. By disregarding wave reflections from the bow shock, that is, for $\lambda = 0$, we obtain Eq. (35), the corrected piston-theory solution, from Eq. (31). Examining Eq. (35) for large K reveals its equivalence to Eq. (37), the solution obtained by applying Miles' first method. Finally, as we make the wedge thinner and thinner, K approaches unity, and we can easily verify that Eq. (35) reduces to the flat-plate solution from piston theory, Eq. (33). (In other words, the shock correction disappears, as it should.) We see that there is a logical framework tying these methods together. Miles' second method, as illustrated by Eq. (38), is suspect because it does not fit into this framework.

Example

In order to illustrate these differences, we choose as an example a rigid wedge oscillating about $x = 1$. Thus $f_w = x - 1$, and Eq. (31) becomes

$$C_{pl} - C_{pu} = 4\tau\bar{\alpha}e^{ikt}(L_1 + iL_2) \quad (39)$$

where

$$L_1 = \left[\frac{2\gamma K^2 - (\gamma-1)}{2 + (\gamma-1)K^2}\right]^{1/2}\left[1 + 2\sum_{n=1}^{\infty}(-\lambda)^n \times \{\cos[k(\Gamma^n - 1)x] - k(\Gamma^n x - 1)\sin[k(\Gamma^n - 1)x]\}\right]$$

$$L_2 = \left[\frac{2\gamma K^2 - (\gamma-1)}{2 + (\gamma-1)K^2}\right]^{1/2}\left[k(x-1) + 2\sum_{n=1}^{\infty}(-\lambda)^n\{\sin[k(\Gamma^n - 1)x] + k(\Gamma^n x - 1)\cos[k(\Gamma^n - 1)x]\}\right] \quad (40)$$

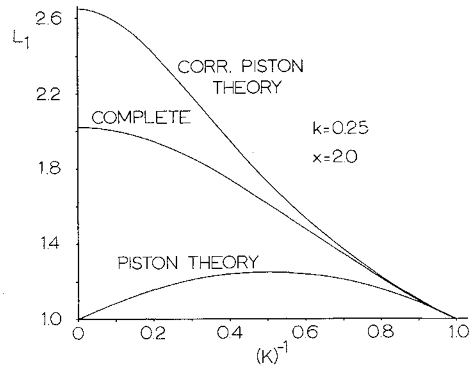

Fig. 4 In-phase component of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

We find for Eq. (32) and Eq. (35), respectively,

$$\left.\begin{aligned} L_1 &= (K)^{-1} + \tfrac{1}{2}(\gamma + 1)b \\ L_2 &= k(x - 1)L_1 \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} L_1 &= \left[\frac{2\gamma K^2 - (\gamma - 1)}{2 + (\gamma - 1)K^2}\right]^{1/2} \\ L_2 &= k(x - 1)L_1 \end{aligned}\right\} \tag{42}$$

Also of interest is the phase angle between the local lift coefficient and the wedge motion, which is given by

$$\Phi = \tan^{-1}(L_2/L_1) \tag{43}$$

The results are presented in Figs. 4–6 for $k = 0.25$ and $x = 2$. The "complete" solution is given by Eqs. (40). Piston theory is given by Eqs. (41), and piston theory corrected for the bow shock is given by Eqs. (42).

We observe that correcting piston theory for the effects of the steady shock wave provides reasonable values for L_1 and L_2 below, say, $K = 2$; piston theory alone diverges from the complete solution rather rapidly. The situation with the phase angle, however, is different. There we find that the phase angle is affected only if the wave reflections are taken into account. Since it has been observed that flutter speeds are not particularly sensitive to variations in the magnitude of the unsteady pressure, it is possible that this change in the phase will be the most important role played by the attenuation factor λ.

Newtonian Theory

A number of authors (see Ref. 13, for example) have shown that the attenuation factor approaches unity as $\gamma - 1$ approaches zero. Therefore we would certainly expect that the differences between corrected piston theory and the complete solution would increase. As a matter of fact, Cole[14] has

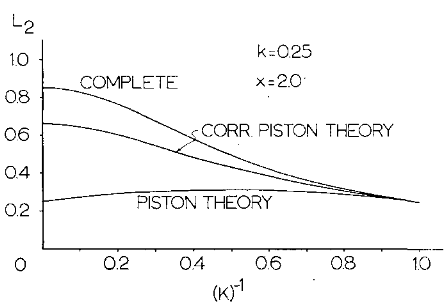

Fig. 5 Out-of-phase component of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

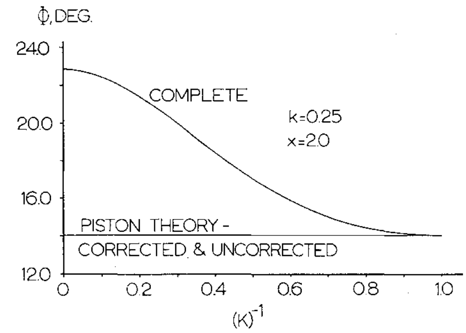

Fig. 6 Phase shift of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

shown that Newtonian theory for slender bodies can be rigorously derived from hypersonic small-disturbance theory. He introduces two additional parameters,

$$\begin{aligned} \epsilon &= (\gamma - 1)/(\gamma + 1) \\ N &= \frac{\gamma + 1}{(\gamma - 1)M_\infty{}^2\delta^2} = \frac{1}{\epsilon M_\infty{}^2\delta^2} \end{aligned} \tag{44}$$

The double limit $\gamma \to 1$, $M_\infty \to \infty$ is implied by requiring that N be fixed and of order unity as ϵ approaches zero. The hypersonic small-disturbance flow quantities are expanded in powers of ϵ, and the transverse coordinate is magnified by dividing by ϵ so that the new coordinate remains of order unity in the limit. Equations and boundary conditions are then derived for the first and second approximations. For our purposes it is only important to note that under these circumstances Newtonian theory can be viewed as a special case of hypersonic small-disturbance theory. Hence we should be able to extract a Newtonian solution from, say, Eq. (31). All that is necessary is to find K, γ, λ, and Γ in terms of N and ϵ.

We choose as an example a rigid wedge oscillating about the vertex, so that $f_w = x$, and Eq. (31) becomes

$$C_{p1} - C_{pu} = 4\tau\bar{\alpha}e^{ikt}\left[\frac{2\gamma K^2 - (\gamma - 1)}{2 + (\gamma - 1)K^2}\right]^{1/2}\left[1 + ikx + 2\sum_{n=1}^{\infty}(-\lambda)^n e^{ik(\Gamma^n - 1)x}(1 + ik\Gamma^n x)\right] \tag{45}$$

We begin by computing $b = \delta/\tau$. A quadratic equation for b is derived from its hypersonic small-disturbance expression, which is

$$b = \frac{2}{\gamma + 1}\left(1 - \frac{1}{K^2}\right) = \frac{2}{\gamma + 1}(1 - b^2H) \tag{46}$$

Here we have written $1/K^2$ as $b^2/M_\infty{}^2\delta^2 = b^2H$. From Eq. (46) we obtain the quadratic equation

$$b^2 + \frac{\gamma + 1}{2H}b - \frac{1}{H} = 0 \tag{47}$$

with the applicable root given as

$$b = -\frac{\gamma + 1}{4H} + \left[\left(\frac{\gamma + 1}{4H}\right)^2 + \frac{1}{H}\right]^{1/2} \tag{48}$$

H and γ are given in terms of N and ϵ from Eqs. (44):

$$\begin{aligned} \tau &= 1 + 2\epsilon + 2\epsilon^2 + 0(\epsilon^3) \\ H &= N\epsilon \end{aligned} \tag{49}$$

Substituting in Eq. (48) and expanding gives

$$b = 1 - (N + 1)\epsilon + N(2N + 3)\epsilon^2 + O(\epsilon^3) \tag{50}$$

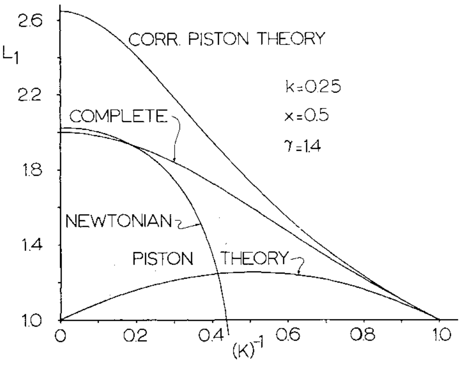

Fig. 7 In-phase component of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

K is thus given as

$$K = M_\infty \delta / b = (N\epsilon)^{-1/2}[1 + (N + 1)\epsilon - (N^2 + N - 1)\epsilon^2 + O(\epsilon^3)] \quad (51)$$

If we now expand the coefficient of the bracketed term in Eq. (45), we find that

$$4\tau\bar{\alpha}e^{ikt}\left[\frac{2\gamma K^2 - (\gamma - 1)}{2 + (\gamma - 1)K^2}\right]^{1/2} = \frac{4\delta\bar{\alpha}e^{ikt}}{[(N + 1)\epsilon]^{1/2}} \times \left[1 + \frac{4N^2 + 8N + 3}{2(N + 1)}\epsilon + 0(\epsilon^2)\right] \quad (52)$$

For small ϵ, this term becomes very large. In this manner we have found in more formal terms that corrected piston theory, as given by Eq. (35), becomes invalid in this limit. This difficulty, we recall, was anticipated in Miles' paper.

The parameters λ and Γ are written in terms of K and γ with the aid of Eqs. (24) and (22). The expansions for K and γ are then introduced, and the resultant expansions for λ and Γ are

$$\lambda = 1 - 4[(N + 1)\epsilon]^{1/2} + 8(N + 1)\epsilon - \frac{2(4N^2 + 10N + 7)}{(N + 1)^{1/2}}\epsilon^{3/2} + O(\epsilon^2)$$
$$\Gamma = 1 - 2[(N + 1)\epsilon]^{1/2} + 2(N + 1)\epsilon - (N + 1)^{-1/2}\epsilon^{3/2} + O(\epsilon)^2 \quad (53)$$

These expressions are then substituted into the general term of the series in Eq. (45). Again we expand in powers of ϵ and find for the summation

$$2\sum_{n=1}^{\infty}(-1)^n\Bigg[1 + ikx - 2n[(N + 1)\epsilon]^{1/2}(2 + 4ikx - k^2x^2) + 2n^2(N + 1)\epsilon(4 + 14ikx - 8k^2x^2 - ik^3x^3) + (N + 1)^{-1/2}\epsilon^{3/2}\Bigg\{n\frac{2(4N^2 + 2N - 5)}{3} - \frac{32}{3}n^3(N + 1)^2 + ikx\left[n\frac{4(4N^2 + 5N - 2)}{3} - \frac{184}{3}n^3(N + 1)^2\right] - k^2x^2\left[n\frac{4N^2 + 8N + 1}{3} - \frac{184}{3}n^3(N + 1)^2\right] + \frac{52}{3}n^3(N + 1)^2ik^3x^3 - \frac{4}{3}n^3(N + 1)^2k^4x^4\Bigg\} + 0(\epsilon^2)\Bigg] \quad (54)$$

We are now faced with summing divergent series of the form

$$\sum_{n=1}^{\infty}(-1)^n n^\nu \qquad \nu = 0, 1, 2, 3, \text{etc.} \quad (55)$$

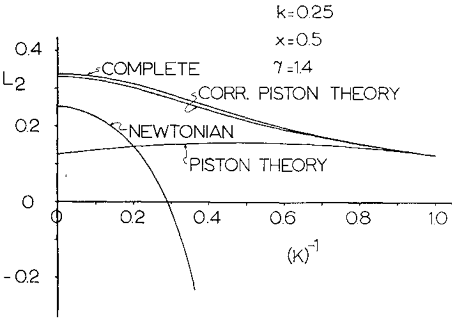

Fig. 8 Out-of-phase component of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

Clearly, this situation is brought about by the sudden prominence of λ and Γ, which has destroyed the convergence of the series. Since the series in its original form from Eq. (45) is convergent, it would obviously have been better to sum it first and then expand the sum in terms of the Newtonian parameters. In the absence of the sum, we are forced to interchange these processes and expand the series term by term before summing. Fortunately, it is not difficult to assign sums to series of the form in Eqs. (55). We refer to a book by Hardy[15] on divergent series where he shows that one may obtain a general result for such sums simply by manipulating the expansion of $(1 - x)^{-1}$ for $x < 1$. Using this result, we find

$$\sum_{n=1}^{\infty}(-1)^n n^\nu = \begin{cases} -\frac{1}{2} & \nu = 0 \\ -\frac{1}{4} & \nu = 1 \\ \frac{1}{8} & \nu = 3 \end{cases}$$
$$\sum_{n=1}^{\infty}(-1)^n n^\nu = 0 \qquad \nu = 2, 4, 6, \text{etc.} \quad (56)$$

Thus we discover that only the leading term plus the terms that contain odd half-powers of ϵ survive in Eq. (54). In addition, the leading term in Eq. (54) just cancels the leading term of the expression in brackets in Eq. (45), leaving as the first term in the sum a term of order $\epsilon^{1/2}$. The subsequent terms, to any order, go in odd half-powers of ϵ. To get the final expansion for $C_{p1} - C_{pu}$, we must multiply this sum by Eq. (52), which has a factor of order $\epsilon^{1/2}$ in the denominator. Hence the product of the two gives an expansion in even powers of ϵ:

$$C_{p1} - C_{pu} = 4\delta\bar{\alpha}e^{ikt}\{2 + 4ikx - k^2x^2 + \epsilon[2 - (10N + 8)ikx + 14(N + 1)k^2x^2 + \tfrac{13}{3}(N + 1)ik^3x^3 - \tfrac{1}{3}(N + 1)k^4x^4] + 0(\epsilon^2)\} \quad (57)$$

This expression is the Newtonian form of Eq. (45). An identical expression can be found by considering the Newto-

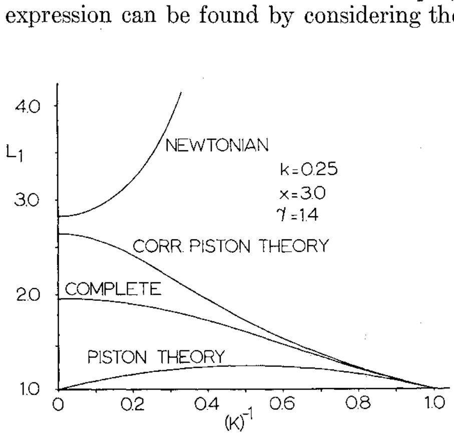

Fig. 9 In-phase component of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

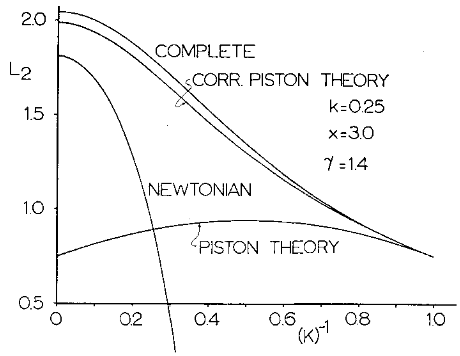

Fig. 10 Out-of-phase component of unsteady lift coefficient vs inverse of hypersonic similarity parameter.

nian differential equations and boundary conditions from Cole's paper. Since Cole's work was done only for steady flow, it is necessary to pose the problem slightly differently by employing the unsteady analogy in reverse and considering a distorted wedge in steady flow.

We present the functions L_1 and L_2 for the rigid wedge oscillating about its vertex in Figs. 7–10. The additional curve in each case is the Newtonian approximation given by Eq. (57). The calculations were done at $x = 1.5$ and $x = 3.0$ to illustrate the rapid deterioration of the Newtonian approximation with increasing x. Actually, it is not really satisfactory anywhere on the wedge, although the situation would certainly improve if $\gamma - 1$ were less than 0.4. On the other hand, the approximation for the steady pressure is better. If we compare the pressure p_0 from Eqs. (17) with its two-term Newtonian approximation, we find that the error ranges from approximately 14% at $K = \infty$ to approximately 20% at $K = 2$. Thus it would appear that the accuracy of the Newtonian approximation in steady flow is not necessarily a basis for predicting its accuracy in unsteady flow. This is probably due to the linearization in the unsteady amplitude which embodies an additional perturbation. Of course, we must admit that the Newtonian representation may not be of practical value anyway because of other phenomena, such as thick boundary layers and detached shock waves, which could cloud the issue.

Conclusions

We see that the bow shock wave plays two roles: first, it alters the state of the gas into which the analogous piston is moving; secondly, it reflects acoustic waves generated by the motion of the piston corresponding to the unsteady wedge motion. In the lower hypersonic range, the latter effect can be ignored. However, as K increases, it becomes significant, most likely because of the ensuing shift in the phase of the unsteady pressure. If a Newtonian representation is desired, Cole's theory can be used, but γ should be less than 1.4 for acceptable accuracy. In this case, reflections from the bow wave are vital, as was illustrated by finding the Newtonian expansion of the solution from ordinary hypersonic flow.

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